No artificial numerical viscosity: from the 1/3 rule to entropy stable approximations of Navier-Stokes equations

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Prologue: Perfect derivatives and conservative differences

$$u_x \quad \mapsto \quad \int_a^b u_x dx = \text{boundary terms of } u \text{ (e.g. } = 0)$$

$$\downarrow$$

$$\frac{u_{\nu+1} - u_{\nu-1}}{2\Delta x} \quad \mapsto \quad \sum \left(\frac{u_{\nu+1} - u_{\nu-1}}{2\Delta x} \right) \Delta x = \text{boundary terms}$$

$$\eta'(u)u_x \quad \mapsto \quad \int_a^b \overbrace{\eta'(u)u_x}^{\eta(u)_x} dx = \text{boundary terms of } u$$

$$\downarrow$$

$$\eta'(u_\nu) \left(\frac{u_{\nu+1} - u_{\nu-1}}{2\Delta x} \right) \quad \mapsto \quad \sum \eta'(u_\nu) \left(\frac{u_{\nu+1} - u_{\nu-1}}{2\Delta x} \right) \Delta x \stackrel{?}{=} \ldots$$

Prologue: Perfect derivatives and conservative differences

$$uu_x \quad \mapsto \quad \int_a^b \overbrace{uu_x}^{\frac{1}{2}(u^2)_x} dx = \text{boundary terms of } u$$

$$\downarrow$$

$$u_\nu \left(\frac{u_{\nu+1} - u_{\nu-1}}{2\Delta x} \right) \quad \mapsto \quad \overbrace{\sum \left(\frac{u_\nu u_{\nu+1} - u_{\nu-1} u_\nu}{2\Delta x} \right)}^{\text{telescoping sum}} \Delta x = \text{boundary terms}$$

- but ...

$$u^3 u_x \quad \mapsto \quad \int_a^b \overbrace{u^3 u_x}^{\frac{1}{4}(u^4)_x} dx = \text{boundary terms of } u$$

$$\downarrow$$

$$u_\nu^3 \left(\frac{u_{\nu+1} - u_{\nu-1}}{2\Delta x} \right) \quad \mapsto \quad \sum \overbrace{\left(\frac{u_\nu^3 u_{\nu+1} - u_{\nu-1} u_\nu^3}{2\Delta x} \right)}^{\text{no perfect deriv.}} \Delta x \mapsto \text{no cancellation}$$

Prologue: Perfect derivatives and conservative differences

$$(\star) \qquad \text{Set} \quad u_x \approx \frac{3}{4\Delta x}\left(\frac{u^4_{\nu+1}-u^4_{\nu}}{u^3_{\nu+1}-u^3_{\nu}} - \frac{u^4_{\nu}-u^4_{\nu-1}}{u^3_{\nu}-u^3_{\nu-1}}\right)$$

$$\sum \frac{3}{4\Delta x}\left(\frac{u^4_{\nu+1}-u^4_{\nu}}{u^3_{\nu+1}-u^3_{\nu}} - \frac{u^4_{\nu}-u^4_{\nu-1}}{u^3_{\nu}-u^3_{\nu-1}}\right)\Delta x \mapsto \text{boundary terms of } u$$

$$\sum \frac{3}{4\Delta x}u^3_{\nu}\left(\frac{u^4_{\nu+1}-u^4_{\nu}}{u^3_{\nu+1}-u^3_{\nu}} - \frac{u^4_{\nu}-u^4_{\nu-1}}{u^3_{\nu}-u^3_{\nu-1}}\right)\Delta x =$$

$$-\sum \frac{3}{4\Delta x}\left(u^3_{\nu+1}-u^3_{\nu}\right)\left(\frac{u^4_{\nu+1}-u^4_{\nu}}{u^3_{\nu+1}-u^3_{\nu}}\right)\Delta x \mapsto \text{boundary terms of } u$$

- How do we come up with $(\star)$?
- Do we have a recipe for general case, $u^3 \mapsto \eta'(u)$?

and ...

- Why do we care about it?

Nonlinear stability - the role of entropy functions

- Euler equations: $$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ m\\ E\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}m\\ qm+p\\ q(E+p)\end{bmatrix}=0$$

$\odot$ Velocity $q = m/\rho$; Pressure $p = (\gamma-1)(E - m^2/2\rho)$;
$\odot$ Specific entropy $S := \ln(p\rho^{-\gamma})$:

$$\overbrace{(-\rho S)}^{\eta(\mathbf{u})}{}_t + \overbrace{(-\rho q S)}^{F(\mathbf{u})}{}_x \begin{cases} = 0 \\ \leq 0 \end{cases}$$

- Nonlinear conservation laws: $\mathbf{u}_t + \nabla_x \cdot \mathbf{f}(\mathbf{u}) = 0$

$$\left\langle \eta_{\mathbf{u}}(\mathbf{u}),\ \mathbf{u}_t + \nabla_x \cdot \mathbf{f}(\mathbf{u}) = 0 \right\rangle \to \overbrace{\eta(\mathbf{u})_t}^{\text{entropy}} + \overbrace{\left\langle \eta_{\mathbf{u}}(\mathbf{u}), \nabla_x \cdot \mathbf{f}(\mathbf{u}) \right\rangle}^{\text{perfect derivatives?}} \begin{cases} = 0 \\ \leq 0 \end{cases}$$

- $\eta(\mathbf{u})$ is an entropy iff $$\boxed{\left\langle \eta_{\mathbf{u}}(\mathbf{u})\ ,\ \nabla_x \cdot \mathbf{f}(\mathbf{u}) \right\rangle = \nabla_x \cdot F(\mathbf{u})}$$

The question of entropy stability

- Entropy conservation: $\eta(\mathbf{u})_t + \nabla_x \cdot F(\mathbf{u}) = 0$

- Entropy decay due to shock discontinuities (Lax):

$$\eta(\mathbf{u})_t + \nabla_x \cdot F(\mathbf{u}) \leq 0$$

- Entropy decay is balanced by perfect derivatives:

$$\int \eta(\mathbf{u}(x, t_2))dx \leq \int \eta(\mathbf{u}(x, t_1))dx, \quad t_2 > t_1$$

- Q. How much entropy decay "$\leq$" is enough?
- A. "physically relevant" entropy decay (w/10 examples):
  - Entropy conservative Euler vs. entropy decay in Navier-Stokes
  - Entropy conservative schemes vs. artificial numerical viscosity
  - Entropy stability and second-order schemes
  - Entropy stability and time discretization
  - Shallow-water eq's: Well-balanced schemes

Entropy conservation: PDEs $\rightarrow$ numerical approximations

$$\Big\langle \eta_{\mathbf{u}}(\mathbf{u}),\ \mathbf{u}_t + \mathbf{f}(\mathbf{u})_x \Big\rangle = 0 \overset{\langle \eta_{\mathbf{u}}(\mathbf{u}),\ \mathbf{f}(\mathbf{u})_x \rangle = F(\mathbf{u})_x}{\Longrightarrow} \eta(\mathbf{u})_t + F(\mathbf{u})_x \begin{cases} = 0 \\ \leq 0 \end{cases}$$

- Semi-discrete approximations:

$$\frac{d}{dt}\mathbf{u}_\nu(t) + \frac{1}{\Delta x}\Big[\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}\Big] = 0 \quad \mathbf{f}_{\nu+\frac{1}{2}} = \mathbf{f}(\mathbf{u}_{\nu-p+1}, \ldots, \mathbf{u}_{\nu+p})$$

- Entropy conservative discretization:

$$\frac{d}{dt}\mathbf{u}_\nu(t) + \frac{\mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}}}{\Delta x} = 0 \overset{?}{\Longrightarrow} \frac{d}{dt}\eta(\mathbf{u}_\nu(t)) + \frac{F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}}{\Delta x} \begin{cases} = 0 \\ \leq 0 \end{cases}$$

Does
$$\boxed{\Big\langle \eta_{\mathbf{u}}(\mathbf{u}_\nu)\ ,\ \mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}} \Big\rangle \overset{?}{\longrightarrow} F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}}$$

so that
$$\Longrightarrow \sum_\nu \eta(\mathbf{u}_\nu(t))\Delta x \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} \sum_\nu \eta(\mathbf{u}_\nu(0))\Delta x$$

A 'faithful' approximation of Euler/Navier-Stokes eq's

Q. How much entropy decay "$\leq$" is enough?

$\star$ Entropy decay in Navier-Stokes equations

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ m\\ E\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}m\\ qm+p\\ q(E+p)\end{bmatrix}=(\lambda+2\mu)\frac{\partial^2}{\partial x^2}\begin{bmatrix}0\\ q\\ q^2/2\end{bmatrix}+\kappa\frac{\partial^2}{\partial x^2}\begin{bmatrix}0\\ 0\\ \theta\end{bmatrix}$$

$$\overbrace{(-\rho S)}^{\eta(\mathbf{u}^\epsilon)}{}_t+\overbrace{(-\rho q S+\epsilon\ln(\theta)_x)}^{F_\epsilon(\mathbf{u}^\epsilon)}{}_x=\overbrace{-(\lambda+2\mu)\frac{(q_x)^2}{\theta}}^{\text{viscosity}}\ \overbrace{-\kappa\frac{|\theta_x|^2}{\theta^2}}^{\text{heat conduction}}\ \leq 0$$

A. Choose $\mathbf{f}_{\nu+\frac{1}{2}}=\mathbf{f}^{*}_{\nu+\frac{1}{2}}$ with no artificial numerical viscosity:

$$\frac{d}{dt}\eta(\mathbf{u}_\nu)+\frac{F_{\nu+\frac{1}{2}}-F_{\nu-\frac{1}{2}}}{\Delta x}$$

$$=\begin{cases}0 & \text{Euler eq's}\\ -\epsilon\Big[\Big(\frac{\Delta q}{\Delta x}\Big)^2\overline{\Big(\frac{1}{\theta}\Big)}+\Big(\frac{\Delta\theta}{\Delta x}\Big)^2\widetilde{\Big(\frac{1}{\theta}\Big)}^2\Big]\leq 0 & \text{NS eq's}\end{cases}$$

Entropy variables and entropy conservative schemes

- Fix an entropy $\eta(\mathbf{u})$. Set Entropy variables: $\mathbf{v} \equiv \mathbf{v}(\mathbf{u}) := \eta_{\mathbf{u}}(\mathbf{u})$.

⊙ Convexity of $\eta(\cdot)$, $\mathbf{u} \leftrightarrow \mathbf{v}$ is 1-1: $\mathbf{v}_\nu = \eta_{\mathbf{u}}(\mathbf{u}_\nu)$

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x}\left[\mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}}\right], \quad \mathbf{f}_{\nu+\frac{1}{2}} = \mathbf{f}(\mathbf{v}_{\nu-p+1}, \ldots, \mathbf{v}_{\nu+p})$$

- Entropy conservation: $\left\langle \eta_{\mathbf{u}}(\mathbf{u}_\nu)\, , \, \mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}} \right\rangle \stackrel{?}{=} F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}$

$$\overbrace{\left\langle \mathbf{v}_\nu\, , \, \mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}} \right\rangle}^{\text{perfect difference}} \quad \text{if and only if} \quad \overbrace{\left\langle \mathbf{v}_{\nu+1} - \mathbf{v}_\nu\, , \, \mathbf{f}^*_{\nu+\frac{1}{2}} \right\rangle}^{\text{perfect difference}} :$$

- Entropy conservative:

$$\boxed{\left\langle \mathbf{v}_{\nu+1} - \mathbf{v}_\nu, \mathbf{f}^*_{\nu+\frac{1}{2}} \right\rangle = \psi(\mathbf{v}_{\nu+1}) - \psi(\mathbf{v}_\nu)}$$

- Entropy flux potential: $\psi(\mathbf{v}) := \left\langle \mathbf{v}\, , \, \mathbf{f}(\mathbf{v}) \right\rangle - F(\mathbf{u}(\mathbf{v}))$

⋆ The scalar case: $\eta'(u)f(u)_x = (\ldots)_x$ - all convex η's are entropies

#1. Scalar examples: $f^{*}_{\nu+\frac{1}{2}} = \dfrac{\psi(v_{\nu+1}) - \psi(v_\nu)}{v_{\nu+1} - v_\nu}$

1.1 Toda flow: $u_t + (e^u)_x = 0$

Exp entropy pair: $(e^u)_t + (e^{2u})_x = 0, \quad \eta(u) = e^u, \ F(u) = e^{2u}/2$

Entropy variable $v(u) = e^u$, potential $\psi(v) := vf - F = \frac{1}{2}v^2$

- Dispersive, entropy-conservative flux:

$$f^{*}_{\nu+\frac{1}{2}} = \frac{\psi(v_{\nu+1}) - \psi(v_\nu)}{v_{\nu+1} - v_\nu} = \frac{\frac{1}{2}v^2_{\nu+1} - \frac{1}{2}v^2_\nu}{v_{\nu+1} - v_\nu} = \frac{1}{2}(v_\nu + v_{\nu+1}) = \frac{1}{2}\left[e^{u_\nu} + e^{u_{\nu+1}}\right]$$

- Entropy-conservative centered scheme (..., Deift, McLaughlin...):

$$\frac{d}{dt}u_\nu(t) = -\frac{e^{u_{\nu+1}(t)} - e^{u_{\nu-1}(t)}}{2\Delta x}$$

$$\frac{d}{dt}e^{u_\nu(t)} = -\frac{e^{u_\nu + u_{\nu+1}} - e^{u_\nu + u_{\nu-1}}}{2\Delta x} \quad \longrightarrow \quad \sum e^{u_\nu(t)}\Delta x = Const.$$

#1. Scalar examples cont'd: $f^{*}_{\nu+\frac{1}{2}} = \dfrac{\psi(v_{\nu+1}) - \psi(v_\nu)}{v_{\nu+1} - v_\nu}$

1.2 Inviscid Burgers: $u_t + (\frac{1}{2}u^2)_x = 0$

quadratic entropy $(\dfrac{1}{2}u^2)_t + (\dfrac{1}{3}u^3)_x = 0, \quad \eta(u) = \dfrac{u^2}{2},\ F(u) = \dfrac{u^3}{3}$

Entropy variable $v(u) = u$, potential $\psi(v) := vf - F = \frac{1}{6}u^3$

- Entropy conservative "$\frac{1}{3}$-rule:

$$\frac{d}{dt}u_\nu(t) = -\frac{2}{3}\left[\frac{u^2_{\nu+1} - u^2_{\nu-1}}{4\Delta x}\right] - \frac{1}{3}\left[u_\nu \frac{u_{\nu+1} - u_{\nu-1}}{2\Delta x}\right]$$

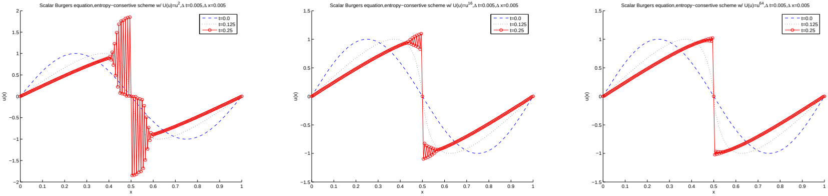
$$\longrightarrow \qquad \sum u^2_\nu(t)dx = Const.$$

- Same with any scalar conservation law and any convex entropy

Conservative differences of u_x <u>and</u> $u^3 u_x = \left(\dfrac{u^4}{4}\right)_x$:

$$\eta(u) = \frac{u^4}{4},\ v = u^3,\ f(u) = u,\ F = \frac{u^4}{4} \text{ and } \psi(u) = \frac{3}{4}u^4 \mapsto f^{*}$$

2 Entropy conservative Burgers' discretization with $\eta_p(u) = u^{2p}$



$p = 1$ $\quad$ $p = 8$ $\quad$ $p = 32$

⊙ The control of L^∞-norm as $p \uparrow$?

- Mesh scale oscillations: entropy as a selection mechanism from micro to macro
- No uniqueness: maximum entropy production principle (E.T. Jaynes, Dafermos, ...)
- Do not "enforce" physically relevant solution by artificial numerical viscosity

#3. Second order artificial numerical viscosity

- Express $\frac{1}{\Delta x}\left(f^*_{\nu+\frac{1}{2}} - f^*_{\nu-\frac{1}{2}}\right)$ as "viscosity" correction

$$\frac{d}{dt}u_\nu(t) = -\overbrace{\left[\frac{f(u_{\nu+1}) - f(u_{\nu-1})}{2\Delta x}\right]}^{\text{centered differencing}} + \frac{1}{2\Delta x}\overbrace{\left[Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}} - Q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}\right]}^{\Delta x^2 (Qu_x)_x}$$

$$f^*_{\nu+\frac{1}{2}} \quad \leftrightarrow \quad Q^*_{\nu+\frac{1}{2}} := \frac{1}{8}\left(\int_{\xi=-1}^{1}(1-\xi^2)f''\big(\underbrace{u_{\nu+\frac{1}{2}}(\xi)}_{u_\nu \to u_{\nu+1}}\big)d\xi\right)\cdot \Delta u_{\nu+\frac{1}{2}}$$

- $Q^*_{\nu+\frac{1}{2}} \sim \Delta u_{\nu+\frac{1}{2}} \longrightarrow$ second-order accuracy
- A comparison principle:
  A difference scheme is entropy stable iff $Q_{\nu+\frac{1}{2}} \geq Q^*_{\nu+\frac{1}{2}}$

3 Artificial numerical viscosity [LxW 1960]

$$Q^{LxW}_{\nu+\frac{1}{2}} = \frac{1}{4}\left[f'(u_{\nu+1}) - f'(u_\nu)\right]^+ \geq Q^*_{\nu+\frac{1}{2}}$$

#4. Beyond second-order accuracy

- Weak formulation of $\mathbf{u}(\mathbf{v})_t + \mathbf{f}(\mathbf{v})_x = 0$:

$$\int_\Omega \left\langle \widehat{w}(x,t)\,,\, \frac{\partial}{\partial t} u(v) \right\rangle dx = \int_\Omega \left\langle \frac{\partial}{\partial x} \widehat{w}(x,t)\,,\, f(u(v)) \right\rangle dx$$

- [Tadmor 1986]

Finite-element discretization: $v \longrightarrow \hat{v}(x,t) = \sum_j v_j(t) \hat{H}_j(x)$:

$$\int_{x_{\nu-1}}^{x_{\nu+1}} \frac{\partial}{\partial x} \hat{H}_\nu(x) f\left(\sum_j v_j(t) \hat{H}_j(x) \right) dx dt = -\left[\mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}} \right]$$

- [LeFloch & Rohde 2000]

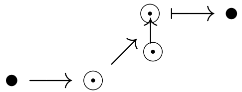
Third-order entropy conservative flux $f^*_{\nu+\frac{1}{2}}$:

4 $$f^{**}_{\nu+\frac{1}{2}} = \int_{-1}^{1} f\left(v_{\nu+\frac{1}{2}}(\xi) \right) d\xi - \frac{1}{12} \left[Q^{**}_{\nu+\frac{3}{2}} \Delta v_{\nu+\frac{3}{2}} - Q^{**}_{\nu-\frac{1}{2}} \Delta v_{\nu-\frac{1}{2}} \right]$$

Systems: $\left\langle \mathbf{v}_{\nu+1} - \mathbf{v}_\nu \,,\, \mathbf{f}^{*}_{\nu+\frac{1}{2}} \right\rangle = \psi(\mathbf{v}_{\nu+1}) - \psi(\mathbf{v}_\nu)$

- Choice of path: N linearly independent directions $\left\{\mathbf{r}^j\right\}_{j=1}^N$

$\odot$ Intermediate states $\left\{\mathbf{v}^j_{\nu+\frac{1}{2}}\right\}_{j=1}^N$:



Starting with $\mathbf{v}^1_{\nu+\frac{1}{2}} = \mathbf{v}_\nu$, and followed by $(\Delta\mathbf{v}_{\nu+\frac{1}{2}} \equiv \mathbf{v}_{\nu+1} - \mathbf{v}_\nu)$

$$\mathbf{v}^{j+1}_{\nu+\frac{1}{2}} = \mathbf{v}^j_{\nu+\frac{1}{2}} + \left\langle \boldsymbol{\ell}^j, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle \mathbf{r}^j, \;\; j = 1, 2, \ldots, N \quad (\mathbf{v}^{N+1}_{\nu+\frac{1}{2}} = \mathbf{v}_{\nu+1})$$

- [Tadmor2003]

The conservative scheme $\dfrac{d}{dt}\mathbf{u}_\nu(t) = -\dfrac{1}{\Delta x}\left[\mathbf{f}^{*}_{\nu+\frac{1}{2}} - \mathbf{f}^{*}_{\nu-\frac{1}{2}}\right]$

$$\boxed{\mathbf{f}^{*}_{\nu+\frac{1}{2}} = \sum_{j=1}^{N} \frac{\psi\left(\mathbf{v}^{j+1}_{\nu+\frac{1}{2}}\right) - \psi\left(\mathbf{v}^{j}_{\nu+\frac{1}{2}}\right)}{\left\langle \boldsymbol{\ell}^j, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle} \boldsymbol{\ell}^j}$$

is entropy conservative: $\left\langle \mathbf{v}_{\nu+1} - \mathbf{v}_\nu \,,\, \mathbf{f}^{*}_{\nu+\frac{1}{2}} \right\rangle = \psi(\mathbf{v}_{\nu+1}) - \psi(\mathbf{v}_\nu)$

#5. Entropy conservative Euler scheme (with W.-G. Zhong)

$$\mathbf{f}^{*}_{\nu+\frac{1}{2}} = \sum_{j=1}^{N} \frac{\psi\left(\mathbf{v}^{j+1}_{\nu+\frac{1}{2}}\right) - \psi\left(\mathbf{v}^{j}_{\nu+\frac{1}{2}}\right)}{\left\langle \boldsymbol{\ell}^j, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle} \boldsymbol{\ell}^j$$

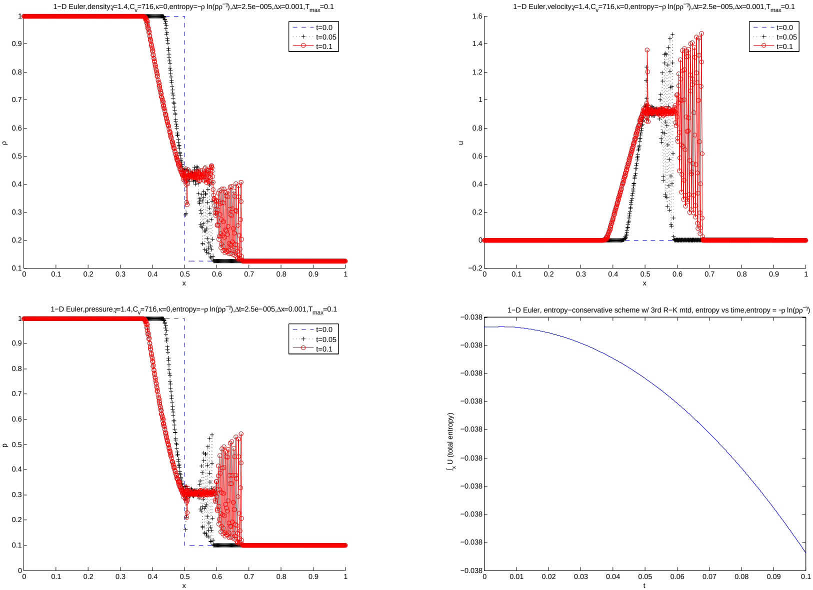
- Entropy function: $\eta(\mathbf{u}) = -\rho S$
- Euler entropy variables: $\mathbf{v}(\mathbf{u}) = \eta_{\mathbf{u}}(\mathbf{u}) = \begin{bmatrix} -E/e - S + \gamma + 1 \\ q/\theta \\ -1/\theta \end{bmatrix}$
- Euler entropy flux potential $\psi(\mathbf{v}) = \langle \mathbf{v}\,, \, \mathbf{f} \rangle - F(\mathbf{u}) = (\gamma - 1)m$
- path in phase-space:
- $\mathbf{v}^0 = \mathbf{v}_\nu, \quad \mathbf{v}^{j+1} = \mathbf{v}^j + \langle \boldsymbol{\ell}^j, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \rangle \mathbf{r}^j, \quad \mathbf{v}^4 = \mathbf{v}_{\nu+1}$

$\left\{\mathbf{r}^j\right\}_{j=1}^3$: three linearly independent directions in $\mathbf{v}$-space (Riemann path)

$\left\{\boldsymbol{\ell}^j\right\}_{j=1}^3$: the corresponding orthogonal system

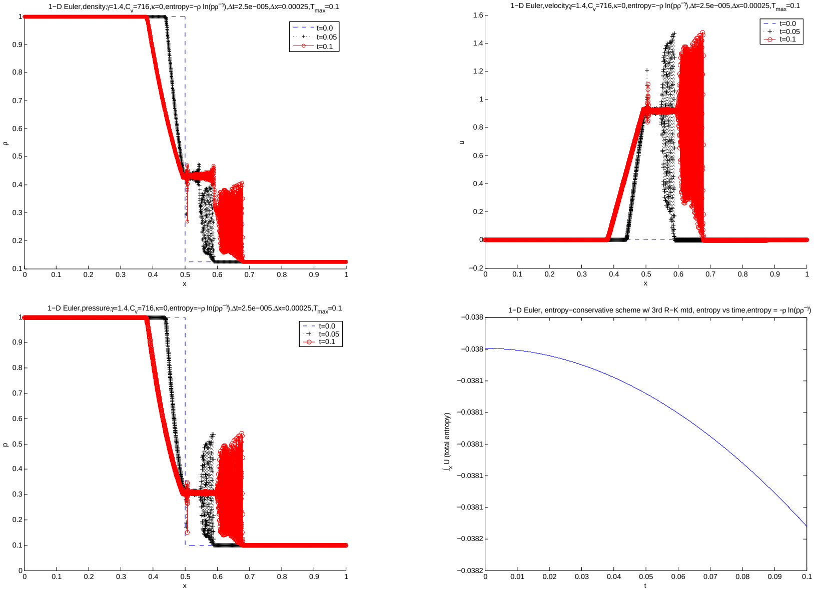
$\left\{m^j\right\}_{j=1}^3$: intermediate values of the momentum along the path

- **5** $\qquad \mathbf{f}^{*}_{\nu+\frac{1}{2}} = (\gamma - 1) \sum_{j=1}^{3} \frac{m^{j+1} - m^j}{\langle \boldsymbol{\ell}^j \,, \, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \rangle} \boldsymbol{\ell}^j$



Entropy conservative results for Euler's Sod problem:

density, velocity, pressure & entropy. 1000 spatial grids, $\eta(\mathbf{u}) = -\rho \ln \left(p\rho^{-\gamma}\right)$



Entropy conservative density, velocity, pressure & entropy w/4000 spatial grids, $\eta(\mathbf{u}) = -\rho \ln \left(p\rho^{-\gamma}\right)$

- Does not "enforce" physical solution with numerical viscosity

#6. Entropy balance in Navier-Stokes eq's

- A semi-discrete scheme of NS equations $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \epsilon \mathbf{d}(\mathbf{u})_{xx}$

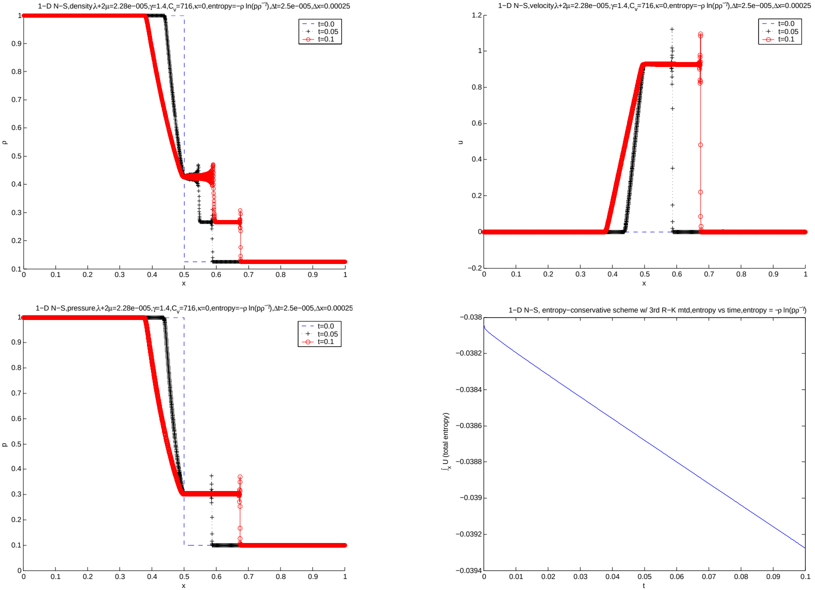
$$\frac{d}{dt}\mathbf{u}_\nu(t) + \frac{1}{\Delta x}\left(\mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}}\right) = \frac{\epsilon}{\Delta x}\left(\frac{\mathbf{d}_{\nu+1} - \mathbf{d}_\nu}{\Delta x} - \frac{\mathbf{d}_\nu - \mathbf{d}_{\nu-1}}{\Delta x}\right)$$

- Entropy-conservative flux $\mathbf{f}^*_{\nu+\frac{1}{2}} = (\gamma - 1)\sum_{j=1}^{3} \frac{m^{j+1} - m^j}{\langle \ell^j , \mathbf{v}_{\nu+1} - \mathbf{v}_\nu \rangle}\ell^j$
- Euler eq's: entropy conservation $\frac{d}{dt}\sum_\nu \eta(\mathbf{u}_\nu(t))\,\Delta x = 0$

6 NS: $\frac{d}{dt}\sum_\nu \eta(\mathbf{u}_\nu(t))\,\Delta x = -\sum_\nu \frac{\epsilon}{\Delta x}\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \frac{\Delta\mathbf{d}_{\nu+\frac{1}{2}}}{\Delta\mathbf{v}_{\nu+\frac{1}{2}}}\Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle \leq 0$

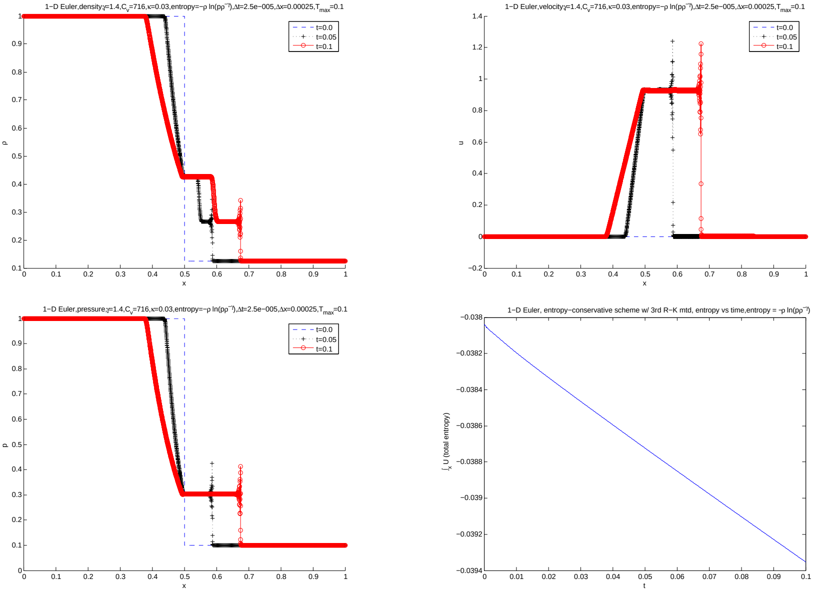
$\star$ $\frac{d}{dt}\sum_\nu(-\rho_\nu S_\nu)\Delta x =$

$$\overbrace{-(\lambda+2\mu)\sum_\nu \left(\frac{\Delta q_{\nu+\frac{1}{2}}}{\Delta x}\right)^2 \overline{\left(\frac{1}{\theta}\right)}_{\nu+\frac{1}{2}}\Delta x}^{viscosity} \overbrace{-\kappa\sum_\nu \left(\frac{\Delta\theta_{\nu+\frac{1}{2}}}{\Delta x}\right)^2 \widetilde{\left(\frac{1}{\theta}\right)}^2_{\nu+\frac{1}{2}}\Delta x}^{heat\ \ conduction} \leq 0$$



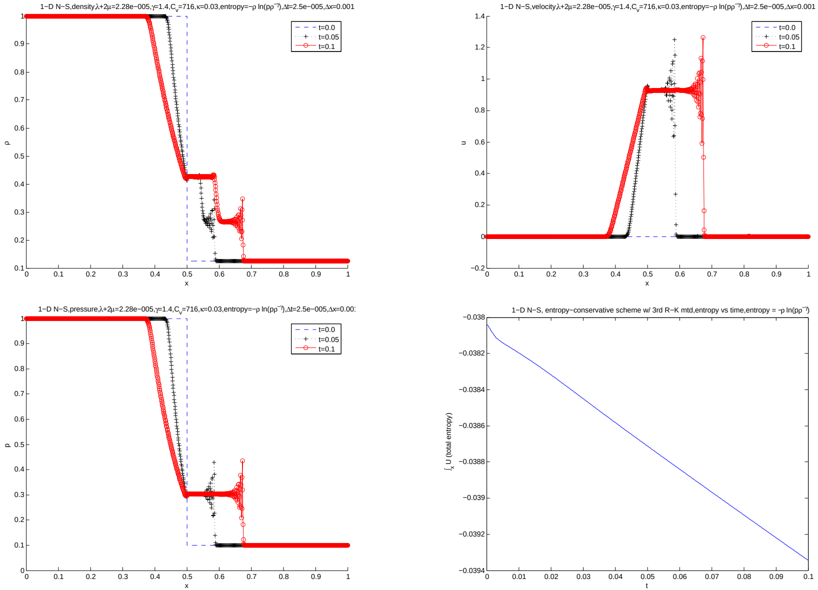
Navier-Stokes equations for Sod's problem:

viscosity but no heat conduction; 4000 spatial grids. $\eta(\mathbf{u}) = -\rho \ln\left(p\rho^{-\gamma}\right)$



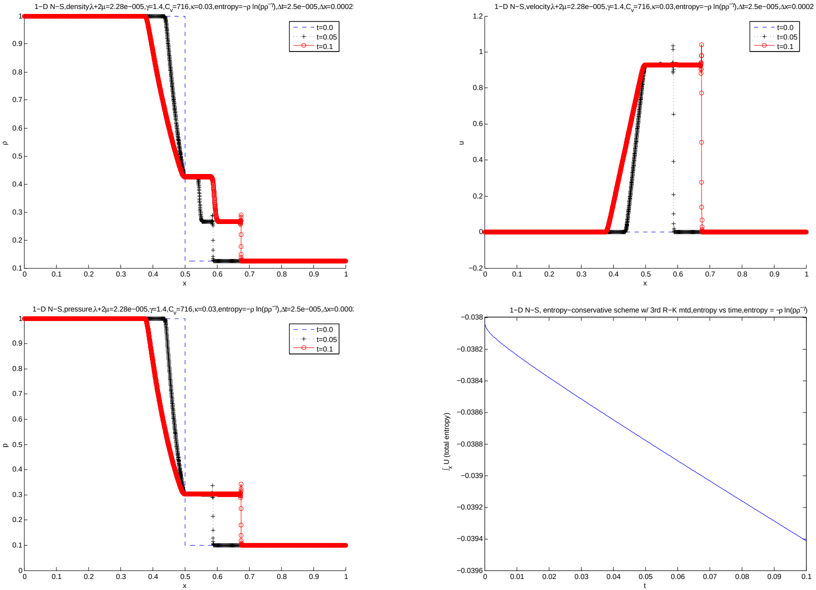
Entropy conservative results for Sod's problem:

heat conduction but no viscosity; 4000 spatial grids. $\eta(\mathbf{u}) = -\rho \ln\left(p\rho^{-\gamma}\right)$



Navier-Stokes equations Sod problem:

both viscosity and heat conduction; 1000 spatial grids, $\eta(\mathbf{u}) = -\rho \ln\left(p\rho^{-\gamma}\right)$



Navier-Stokes equations Sod problem:

viscosity and heat conduction; 4000 spatial grids, $\eta(\mathbf{u}) = -\rho \ln\left(p\rho^{-\gamma}\right)$

#7. Entropy stability for fully discrete schemes

7.1 The backward Euler scheme - unconditional stability

$$\mathbf{u}_\nu^{n+1} = \mathbf{u}_\nu^n - \lambda\Big[\mathbf{f}^*_{\nu+\frac{1}{2}}(\mathbf{v}^{n+1}) - \mathbf{f}^*_{\nu-\frac{1}{2}}(\mathbf{v}^{n+1})\Big], \quad \mathbf{v}^{n+1} = \mathbf{v}(\mathbf{u}(t^{n+1}))$$

7.2 Crank-Nicolson: $\overline{\mathbf{v}}^{n+\frac{1}{2}} := \int_{-1}^{1} \mathbf{v}\big(\mathbf{u}^{n+\frac{1}{2}}(\xi)\big)\,d\xi \not\approx \mathbf{u}^{n+\frac{1}{2}}$

$$\mathbf{u}_\nu^{n+1} = \mathbf{u}_\nu^n - \lambda\Big[\mathbf{f}^*_{\nu+\frac{1}{2}}\big(\overline{\mathbf{v}}^{n+\frac{1}{2}}\big) - \mathbf{f}^*_{\nu-\frac{1}{2}}\big(\overline{\mathbf{v}}^{n+\frac{1}{2}}\big)\Big]$$

is an entropy stable (– conservative) scheme iff the semi-discrete is

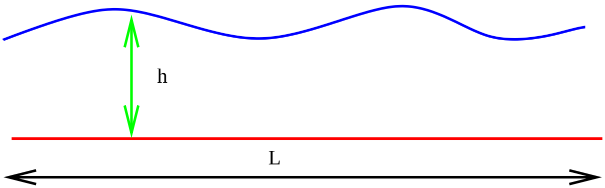
7.3 Modify LxF:

$$\mathbf{u}_\nu^{n+1} = \frac{1}{4}\big(\mathbf{u}_{\nu+1}^n + 2\mathbf{u}_\nu^n + \mathbf{u}_{\nu-1}^n\big) + \frac{\lambda}{2}\big[\mathbf{f}\big(\mathbf{u}_{\nu+1}^n\big) - \mathbf{f}\big(\mathbf{u}_{\nu-1}^n\big)\big]$$

$$Q^{LxF}_{\nu+\frac{1}{2}} = \frac{\Delta x}{2\Delta t} I_{N\times N} \geq Q^* \longrightarrow \text{CFL}: \ \frac{\Delta t}{\Delta x}\max_{\lambda}|\lambda(A+Q^*)| \leq \frac{\sqrt{2}-1}{2}$$

2D shallow water equations (with U. Fjordholm, S. Mishra)

- Vertical scale of motion $\ll$ Horizontal scales



- Navier-Stokes equations $\mapsto$ Shallow water equations.

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) + \partial_y \mathbf{g}(\mathbf{u}) = \kappa \partial_x \left(h \partial_x \mathbf{d}(\mathbf{u}) \right) + \kappa \partial_y \left(h \partial_y \mathbf{d}(\mathbf{u}) \right),$$

- Conserved variables: $\mathbf{u} = [h,\ uh,\ vh]^\top$
  height h and velocity field (u, v);
- Convective fluxes:
  $\mathbf{f} = [uh,\ u^2 h + gh^2/2,\ uvh]^\top, \quad \mathbf{g} = [vh,\ uvh,\ v^2 h + gh^2/2]^\top$
- Viscous fluxes $\mathbf{d} = [0,\ u,\ v]^\top$

2D shallow-water equations cont'd

$$
\begin{aligned}
h_t + (hu)_x + (hv)_y &= 0,\\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= \kappa\big((hu_x)_x + (hu_y)_y\big),\\
(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= \kappa\big((hv_x)_x + (hv_y)_y\big),
\end{aligned}
$$

- $\kappa > 0$ – constant eddy viscosity;
  determines transfer of energy to small scales
- Entropy - total energy $\eta(\mathbf{u}) = (gh^2 + u^2h + v^2h)/2$
  dissipated by eddy viscosity
- Quadratic fluxes in $h, \sqrt{h}u$ and $\sqrt{h}v$:

#8. Explicit Energy conservative (EEC) fluxes

- Energy conservative fluxes - an algebraic approach to ...

$$\left\langle \mathbf{v}_{\nu+1,\mu} - \mathbf{v}_{\nu,\mu}\, ,\, \mathbf{f}^{*}(\mathbf{u})_{\nu+\frac{1}{2},\mu} \right\rangle = \psi(\mathbf{v}_{\nu+1,\mu}) - \psi(\mathbf{v}_{\nu,\mu})$$

- ... using the average values: $\overline{w}_{\nu+\frac{1}{2}} := \frac{1}{2}\left(w_{\nu} + w_{\nu+1}\right)$

8 $$\mathbf{f}^{*}(\mathbf{u})_{\nu+\frac{1}{2},\mu} = \begin{bmatrix} \overline{h}_{\nu+\frac{1}{2},\mu}\overline{u}_{\nu+\frac{1}{2},\mu} \\ \overline{h}_{\nu+\frac{1}{2},\mu}\left(\overline{u}_{\nu+\frac{1}{2},\mu}\right)^2 + \frac{g}{2}\left(\overline{h^2}\right)_{\nu+\frac{1}{2},\mu} \\ \overline{h}_{\nu+\frac{1}{2},\mu}\overline{u}_{\nu+\frac{1}{2},\mu}\overline{v}_{\nu+\frac{1}{2},\mu} \end{bmatrix}$$

$$\mathbf{g}^{*}(\mathbf{u})_{\nu,\mu+\frac{1}{2}} = \ldots$$

- Energy conserving fluxes $\longrightarrow$ path in phase space?

#9. Energy stable scheme - Roe-type viscosity

9 $$\mathbf{f}^{ERoe}(\mathbf{u})_{\nu+\frac{1}{2},\mu} = \mathbf{f}^{*}(\mathbf{u})_{\nu+\frac{1}{2},\mu} - \frac{1}{2} R_{\nu+\frac{1}{2},\mu} |\Lambda| R^{\top}_{\nu+\frac{1}{2},\mu} \Delta \mathbf{v}_{\nu+\frac{1}{2},\mu}$$

$$\mathbf{g}^{ERoe}(\mathbf{u})_{\nu,\mu+\frac{1}{2}} = \ldots$$

#10. Well-balanced shallow-water schemes

$$
\begin{aligned}
h_t + (hu)_x + (hv)_y &= 0\\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= -ghb_x\\
(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= -ghb_y
\end{aligned}
$$

- Steady solutions due to bottom topography $b(x,y)$:
  - ⊙ Lake at rest: $H := h + b = Const.; \quad u = v = 0$
  - ⊙ Other equilibrium states: $uu_x + vu_y + gH_x = 0$
    $uv_x + vv_y + gH_y = 0$

10. Recovered by the energy conserving – $E(\mathbf{u}) = \frac{1}{2}\left(h(u^2+v^2) + ghH\right)$:

$$
E(\mathbf{u})_t + \overbrace{\frac{1}{2}\left(hu^3 + huv^2 + ghH\right)_x}^{F(\mathbf{u})_x} + \overbrace{\frac{1}{2}\left(hu^2v + hv^3 + ghH\right)_y}^{G(\mathbf{u})_y} \left\{ \begin{array}{c} = \\ \leq \end{array} \right. 0
$$

Perturbation of lake at rest

Other equilibrium steady state of shallow-water

$$uu_x + vu_y + gH_x = 0$$

$$uv_x + vv_y + gH_y = 0$$

$$E(\mathbf{u})_t + \overbrace{\frac{1}{2}\left(hu^3 + huv^2 + ghH\right)_x}^{F(\mathbf{u})_x} + \overbrace{\frac{1}{2}\left(hu^2v + hv^3 + ghH\right)_y}^{G(\mathbf{u})_y} \begin{cases} = \\ \leq \end{cases} 0$$

• "Entropy stability theory for difference approximations of nonlinear conservation laws ...", Acta Numerica 2003.

• "Energy-preserving and stable approximations for the 2D SW eqs", Proc. of the Third Abel Symposium, 2008.

• "Energy preserving and energy stable schemes for the SW eqs", London Math. Soc. Lecture Notes #363, 2009.

• "Arbitrarily high order accurate entropy stable ENO schemes for systems of conservation laws", SINUM 2012.

THANK YOU